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–Introduction–

About this Book

Why this curriculum?

Previous to when I began teaching at Shining Mountain Waldorf School in 1994, my experiences as a math teacher had been fairly typical. I was to teach out of some textbook, and it was expected that my class would complete a certain portion of that book. Perhaps I had had some freedom in deciding which topics to leave out, but I certainly wasn't expected to create a mathematics curriculum.

When I arrived at Shining Mountain and was given the task of teaching middle school math, I was surprised to find that there were no textbooks, and that very little existed in terms of a math curriculum guideline. I was expected to create my own materials. This newfound freedom was both exciting and daunting. I proceeded to research what was happening at other Waldorf schools regarding math curriculum. I spent a great deal of time creating my own worksheets. My original question, "What should I teach?", soon became transformed into, "What topics would best help in developing the thinking and imagination of my middle school students?"

This book is intended to share with others what I have discovered and developed during this exciting journey.

New in the third edition!

I have reworked and reworded many sections. For those people who have the second edition and use it regularly, I would recommend buying this edition because the changes are significant enough. You may also download the major changes from wholespiritpress.com, although the numerous minor changes are not included there. The major changes in this edition are:

- A reworking of the seventh grade *percents* unit.
- A major reworking of the seventh grade *ratios* unit.
- A reworking of the seventh grade *percents* unit.
- A new section, *Calculating the Area of Four Types of Triangles*, for eighth grade.
- A new section, *Calculating the volumes of a Cone and a Pyramid,* for eighth grade.
- A reworking of *The Square Root Algorithm- without zeroes* (written as a computer program), with a step-by-step example (really, it makes sense this time!) for eighth grade.
- An addition to the appendix: *Table of Square Roots*.

Who is this for?

While most of my teaching experience is within the Waldorf school system, the curriculum presented here can be effectively used by any teacher wishing to bring meaningful, age-appropriate material to their students. The explanations in this book are also useful for parents (or tutors) who are helping their children in a math class that uses the curriculum laid out in this book.

While most of the material is not overly difficult, much of it is foreign even for math majors. In writing this book, I have assumed that the reader may be weak in math, so, with some effort, even the most difficult topic should be understandable. Keep in mind that the teacher who has to struggle to master a subject, will be all the better at teaching it. I have often been at my best when presenting material that I initially found difficult to grasp.

Caution! This is only a guideline.

There are many ways to achieve the same goals. What I have presented in this curriculum guideline is what I have found works for me, at my school, with my students. Some of the topics may not be appropriate for you or your students. Bear in mind also that this particular curriculum guideline is constantly being modified. Teachers must decide for themselves which topics they can best present in the classroom.

Multiple solutions to math problems

With most math problems there are a variety of methods to arrive at the correct answer. The teacher is then faced with the decision of which method to bring into the classroom, and whether it is beneficial to teach more than one method. This depends largely upon the learning styles of the individual students – some students find it confusing to see different methods, but for many it is helpful to see more than one way to find a solution; it develops flexibility in their thinking. Of course, it is always good to encourage students to devise their own methods.

In this book, I mostly describe only one method to solve a given problem. Please keep in mind that there are usually other methods that I haven't mentioned, and your method may be just as effective, or more effective, than mine.

You can't cover all this!

I have never, with any class, covered all the material listed in this book. Which topics the teacher can cover during the year will depend upon the teacher and the class, as well as the amount of time available. This curriculum assumes a well-prepared class coming into sixth grade. It also assumes that track classes meet three times per week in the seventh and eighth grades, and twice per week in the sixth grade. Ideally, all three grades should have two math main lessons during the year. Not all schools will find this possible. Obviously, it is important not to rush through the material for the sake of, "getting through the curriculum." I have noticed over the years that I teach fewer and fewer topics, allowing more time, and depth, for each topic.

Main Lessons and Track Classes

Some of the material presented here assumes familiarity with the Waldorf lesson structuring, known as main lessons and track classes. Track classes are basically the norm in most public schools. A math track class is typically 40 minutes long and meets three times per week for the whole year, while studying a variety of topics. The concept of a main lesson is more unique to Waldorf schools. A main lesson meets first thing in the morning for up to two hours, everyday for a period of 2½ to four weeks. It usually concentrates on just one particular topic. In Waldorf schools, we typically introduce a completely new topic in the main lesson (e.g., algebra in seventh grade), and then work on developing skills in the track classes.

The order of topics

The topics in this book are arranged by grade according to subject area. It is not intended that a teacher should cover the topics in the order that they appear in this book! In every grade, I have listed all the arithmetic topics first, then the algebra topics, and lastly the geometry topics. In many cases, it is necessary to do things in a completely different order (e.g., first cover some arithmetic, then some geometry, and then something else). Teachers using my workbooks will likely find it best to cover topics in the order that they appear in the workbook. *Don't forget that some topics don't appear in the workbooks at all!*

"Making Math Meaningful" workbooks

In order to facilitate bringing this curriculum to life, I have created workbooks (one for each of the grades 6 through 8) from the topics found in this curriculum guide. Each workbook is essentially intended to be the homework assignments given during the year in a math track class. Keep in mind that the workbooks mostly do not include material that ought to be taught during main lesson. Also, there are other topics (e.g., puzzle problems) that appear in this curriculum guide but are not included in the workbooks. These workbooks can be purchased through Whole Spirit Press (www.wholespiritpress.com; 303-979-5820).

Work in progress

I am currently working on more books in the *Making Math Meaningful* series, particularly one on first through fifth grade math, and a variety of things for high school math. Contact Whole Spirit Press to find out if anything new is available.

Questions, Comments, or Feedback?

Please contact Whole Spirit Press if you have any questions, comments, or feedback.

Some Thoughts on Teaching Math

The state of mathematics education today

(The following bit of historical background may help you to understand where mathematics education stands today.)

Pythagoras called himself a philosopher, which he said meant a "seeker of truth." Today, we consider him to be the first one. Up until the beginning of the 18th century, all the greatest mathematicians were philosophers in the broadest sense. They wouldn't have been called just "mathematicians" for they sought knowledge in many realms. They studied the multiple branches of science. They were fluent in several languages. They were perhaps even poets or artists, and also deeply spiritual people. Yet, perhaps out of necessity, all this changed. Fields of study became specialized. Soon there were people called mathematicians whose focus became increasingly narrow, who were disconnected from the other disciplines.

In the latter half of the 19th century and the first half of the 20th century there was something of a revolution in the world of mathematics. It began with the collapse of Euclidean geometry, which, until then, had been thought of as the perfect model of scientific/mathematical thought. For more than 2000 years, Euclid's 13-volume work *The Elements* had been regarded as absolute truth. One of his basic assumptions was (loosely stated) that two parallel lines never meet. But in the late 1800's, mathematicians suddenly realized that this assumption was *not necessarily true*. All of Euclid's work was called into question. Mathematicians then embarked on a search for the ultimate logically sound basis for all mathematics. The study of mathematics became what it largely is today: the study of formal math and logic. Even the notion of truth in math was doubted. Bertrand Russell said at the time, "math is the subject in which we never know what we are talking about, nor whether what we are saying is true."

Mathematicians spent the next few decades trying to create the perfect mathematical system upon which all mathematics could happily rest. Then, in 1931, the whole endeavor came to a sudden halt when Kurt Gödel proved that such a system could not actually exist. Mathematics was thrown into chaos. In the next two decades the logical positivists emerged, asserting that we can only come to knowledge through our physical senses. According to the logical positivists:

- The purpose of mathematics is to start with *meaningless terms* and then to prove things about them.
- The only real math is formal math (e.g., proofs).
- Math is only meaningful when applied to the sense world.
- Anything to do with imagination or intuition is meaningless.

The assertions of the logical positivists had a profound effect on the teaching of math right down to the kindergarten level. Math was stripped of its meaning.

It was in this climate that the Soviets launched the first satellite (Sputnik) in 1957. Suddenly, America became focused on beating the Soviets, and the role of mathematics education was to help produce more scientists in the effort to win this race. Through this came "new math", which had the objective of teaching as much high-level math as possible at as young of an age as possible. Set theory (e.g., union and intersection of sets) is one example of this. New math has recently fallen out of favor, but still, many schools list their probability and statistics curriculum as starting in kindergarten. There is now a realization that mathematics education needs to be overhauled, and there are numerous new models out there that are attempting to redefine the subject.

In the meantime, mathematics education is further challenged by an overemphasis on testing and a constant pressure to quickly get through an unrealistic amount of material. There is little room left for depth, contemplation, self-discovery, or flexibility.

The end result is that mathematics has become rather meaningless for most students. I believe the only way to save mathematics education is by making it *meaningful*.

How can we make math meaningful?

While many people may agree that math, as it is taught today, is mostly meaningless, there is not much agreement on how to give it meaning. Those who see math only as a necessary preparation for another subject like engineering or economics, argue that making math meaningful means teaching practical math. While I agree that we need to teach useful and practical topics like percents and algebra, the most successful topics that I teach seem ostensibly *useless*. In fact, my favorite topics include *converting repeating decimals to fractions* (6th grade), the *square root algorithm* (7th grade), and *stereometry* (8th grade). I had never heard of any of these before I taught in a Waldorf school. Of course, while these topics are useless in a practical sense, they are very useful in helping to develop students' thinking.

So, what can make math meaningful for our students? Here are some ideas:

- **Make it developmentally appropriate.** As with teaching reading, the question of when to introduce a math topic should not be: "Are the children able to learn this material now?" The question should be, rather: "Are the children developmentally ripe for this material?" For this reason, The topic of probability should wait until ninth grade, the bulk of algebra should wait until ninth grade, and volumes should wait until eighth grade – just to name a few examples.
- **Work with questions.** Does the topic at hand answer a real question that lives within the student? It is always good to write a question on the board that is quite difficult and then to tell the class that we will be working to find an answer over the next few weeks or even months. It is quite meaningful for the students when the class is finally able to answer that seemingly impossible question that was posed so long ago.
- **Allow for Depth.** The tendency in the mainstream is to teach too many topics. This means there is not enough time to cover anything in detail. Avoid rushing through things, and see each topic through to its proper completion. By covering a topic in depth, we allow thinking to develop more effectively.
- **Challenge the students.** If the material is appropriately difficult, the student may well reach a point of frustration. Overcoming that frustration, finding a solution, and coming to an understanding of something that was previously confusing, is a very meaningful process for the student. The challenge for the teacher is to make sure that the student completes this process.
- **Offer interesting material.** Sometimes material can be made more interesting when students can see how it relates to the real world, (e.g., 6^{th} grade business math). At other times, students are quite fascinated by things like calculating 2 to the 100^{th} (7^{th} grade), or figuring out how arithmetic would have been different if people had had eight fingers instead of 10 (8^{th} grade number bases).
- **Provide the historical context.** The more historical context that can be woven into the math class, the better. It makes it quite meaningful for the students to realize that they are living and breathing the same thoughts that the greatest minds in history also have struggled with.

Education for social change

Be it social issues, environmental issues, or political issues, the world today is facing many challenges. The causes of many of our crises include, in my opinion, our disconnection from nature and the spiritual world, lack of community, lack of meaning in people's lives, and lack of independent thinking. Noam Chomsky quite bluntly attributes many of these problems to the fact that people today have become ignorant (lack of ability to think), apathetic (lack of caring), and passive (lack of will). Waldorf education combats this by working consciously to develop healthy human beings who are capable of thinking creatively and independently, who have a love of the world, and who have the courage and will to act.

You may ask: "What does this have to do with math?" I used to believe that the purpose of math was simply to develop the students' thinking. I now see math as a means to develop the *whole human being* – not just thinking, but also feeling and willing as well. Good math students have not only developed their ability to think, but, in the feeling realm, they find joy in learning and are able to overcome frustration, and, in the willing realm, they are determined, hard workers.

It is clear to me that developing the whole human being is the key to solving many of the problems in our world today. Mathematics education is part of that noble effort.

What makes a good math student – struggling is valuable!

I remember thinking, years ago, that many students don't have the ability to develop into good math students. My thoughts on this have changed radically over the years. I now believe that all but the rare student has the cognitive ability to "get" even the most difficult math that is presented at school. In my opinion, there are other factors that play a more decisive role than ability in determining whether a given student will develop into a good math student.

These key factors are: *organization, determination, and the ability to overcome frustration*.

In light of this, much of what I do focuses on the development of these characteristics.

I am fond of telling my students that learning math should be hard. If it were always easy, then it wouldn't be as valuable to learn. What do students do when they encounter a difficult math problem? The problem may appear impossible and the student may have little idea of what to do. They may make mistakes. That's fine! It is an incredibly valuable process for the students to work through their frustration and confusion. However, this is in contrast to today's culture of fast service and high-speed information. Yet, it's still an excellent lesson in life. Our job as teachers is largely to encourage our students to work through their struggles. Likewise, it is also valuable for us, as teachers, to struggle with math.

The two exceptions to this rule "good math is hard math" are: algebra and word problems. With these two topics, we should keep it simple. Our goal with algebra is to have our students enter high school thinking that they are good at algebra. With word problems, our goal is to have the students enter high school feeling that word problems are either fun and interesting, or at least manageable.

General goals

Besides developing the students' competence in the specific subject matter listed in this curriculum guide, middle school mathematics also works on building their general cognitive skills. Mental math works on mental quickness, memory and concentration. Geometry works on spatial imagination and visualization. Keeping notebooks that are divided into sections helps to develop crucial organization skills, as does writing homework problems in a neat, sequential manner, so that the work can easily be followed. Integrating history into the math lesson cultivates an awareness of how all subjects are interrelated. Review is important in order to maintain solid skills. Forgetting is part of learning! Overall, keep in mind that a successful middle school math program has students entering high school with *solid basic skills*, a *healthy imagination*, and *enthusiasm for learning*.

Geometry and the imagination

Einstein said, "Imagination is more important than knowledge. Knowledge is limited. Imagination encircles the world." Imagination is what gives rise to judgment and the ability to develop one's own morality and ethical basis. Lack of imagination is reflected in people's inability to think for themselves, develop their own opinion, and question the "facts" presented in the media. The lack of values in today's culture, I believe, is rooted in our weak imagination. A healthy imagination leads to an ability to solve problems – to *imagine* creative solutions. We need leaders that can *imagine* solutions to our world's problems.

Imagination is woven throughout Waldorf education, and, in math, this is most apparent in how we teach geometry. It is hardly surprising that in today's culture, where imagination is viewed as something childish, flaky, and unfocused, that geometry has been stripped down to a study of measurement (perimeter, area, volume), formulas, and proofs. (Remember tenth grade two-column proofs?) Since discovering Waldorf education, I have learned that there is a whole other realm of geometry that I never knew existed.

I call this "pure geometry". Pure geometry is the study of form, purely for its own sake (not just for the sake of measurement and performing calculations, etc.). While I received none of it in my schooling, it receives a wonderful emphasis in the Waldorf curriculum through these subjects: form drawing (grades 1-5), eurythmy (grades K-12), clay modeling (grades K and up), geometric drawing (grades 6 and 7), stereometry (8^{th} grade), loci (8^{th} grade), descriptive geometry (9^{th} grade), and projective geometry (11^{th} grade).

Pure geometry deals with the *exact imagination*, which allows a person to fully and accurately picture something in their head. In the lower grades, this is cultivated by listening to stories. In geometry, the exact imagination is cultivated by picturing forms – two or three-dimensional. The following quote from Rudolf Steiner regarding math has made the greatest impact on me in this regard:

> *"Whenever possible, try to have the students picture geometric forms in movement."*

This very process enlivens the imagination in an exact and exciting way. The idea is to be able to clearly picture a form going through a process of transformation. Students, for the most part, find this method very satisfying, and I have used this theme repeatedly in this curriculum. Examples are the *shear and stretch* (7^{th} grade), the proof of the formula for the area of a circle (8^{th} grade), the transformation of solids (8^{th} grade, *stereometry*), and the transformation of curves (8^{th} grade, *loci*).

Separation of form and number

For me, it is helpful to think of *form* (pure geometry) as an aspect of the physical world, and to think of *pure number* as something that comes from the spiritual world. Unfortunately, today, *form* and *number* are usually blended together. In mainstream education, geometric shapes are analyzed by measuring them and then calculating such things as the area and perimeter. In high school, the study of geometry is the main platform from which to study logic through deductive proofs. And Cartesian geometry allows us to attach equations to practically any shape, thereby reducing geometric shapes to formulas. While most of this should be taught, there is a fundamental problem when the student experiences only this and not any pure geometry – geometry freed from number, measurement, and formulas.

But, just as geometry should be studied as much as possible without getting into number and formula, topics involving number should be studied without geometric pictures. Many Waldorf schools do geometry quite well, but often the teaching of what should be pure number gets muddled because of the introduction of pictures. Teachers use pictures as a crutch to help the students learn particular concepts. Students are led to think that addition is counting beans, that fractions are pizza, and that percents are how much water is in a glass. Introducing a pure number topic (e.g., addition, fractions or percents) by using imaginative pictures may make it easier for a student to seemingly comprehend a topic in the short-term, but it does not capture the essence of the topic, and can become a barrier to reaching a deeper understanding later.

As teachers, we must be careful to ensure that topics of pure number such as arithmetic, fractions, percents, etc., are developed in the child's mind free from physical pictures.

Fractions aren't pizza, but later we can point out that dividing a pizza is one way to see a fraction. Addition isn't just counting beans. Similarly, we can talk about the percentage of water in a glass, but only once the students have sufficiently grasped the idea that percents represent a fractional part of a hundred.

Algebra

Although the content of the algebra in this curriculum may not be substantially different from the norm, there is one crucial difference in the way that I teach algebra: I present the basic fundamentals of algebra in a three-week main lesson block in seventh grade. After this, the subject is "put to sleep" for up to a year, allowing the child to digest this important step before building on it. This is in direct contrast to the normal approach to algebra, which gives an initial introduction to algebra, and then immediately builds on this not-yet-firm foundation.

The algebra curriculum during the middle school years should not have priority over other material. Indeed, it is not completely necessary to cover many of the topics listed here under eighth grade *algebra*, by the end of the eighth grade year. Some eighth grade teachers may need to dedicate more time to reinforcing skills from the earlier years, or may need to cover material that had been previously left out. At the very least, it is necessary to review the algebra covered in seventh grade during the eighth grade year. However, any algebra topic that is not covered in eighth grade will most likely be covered as part of any standard ninth grade algebra course. In contrast, most of the topics listed here under *geometry* and *arithmetic* should be covered by the end of eighth grade, as it is generally not part of high school curricula.

Avoid the temptation of doing a lot of algebra because it is more familiar to you, or because, perhaps unconsciously, you are trying to prove to parents or fellow colleagues that your class is quite advanced in math since they have done so much algebra. I have seen this happen all too often. The truth, in my opinion, is that *the bulk of algebra belongs in ninth grade, when the child is most developmentally ripe for it*. The goal is to have our students enter high school feeling confident about their ability to do algebra.

Word problems

In the last few decades, the mainstream has placed much emphasis on problem solving and word problems. My experience is that the vast majority of students learn to hate word problems by the time they enter high school.

I feel that real algebraic word problems belong in eleventh grade. (An example of an algebraic word problem is: "Find the dimensions of a rectangle that has an area of 28 square feet and a perimeter of 21 feet?" These kinds of word problems require analytical thinking, and the ability to see the equations behind the words.) Any work on word problems in middle school needs to be simple. This is the exception to my above rule that "good math is hard math." An approach that I use in middle school is to include one or two word problems on each homework sheet. The goal is to have the students enter high school feeling that word problems are either fun and interesting, or at least manageable.

Group Work

Perhaps the biggest change in my teaching that has occurred in the last few years is that I now try to have the students work together in small groups during math class. I assign the groups carefully, and have them try to help each other with problems that they got wrong on homework, and to answer other questions. Often I have groups work together on a challenging or new problem that would be too difficult for most students to work on independently. My eighth grade workbook has sheets that are specifically designed to guide the students to new concepts in a group rather than being "fed" these concepts by the teacher. An emphasis on group work can take more time, but it can allow students to benefit in ways that wouldn't otherwise be possible. All this being said, it still remains vital for the class as a whole to be able to sit and quietly focus as the teacher guides them through a difficult problem or concept.

Mental math

When I was in school in the 70's, mental math (doing calculations in your head) was something we did if there was time left before the end of class – and there usually wasn't. Mental math is now a big part of what I do. I have seen mental math used as a springboard for developing a student's confidence in their math skills, and confidence in their thinking. This alone is a good argument for why calculators shouldn't be used regularly until tenth grade.

Many teachers fade out the mental math as they enter middle school. This is unfortunate. I feel that the peak of mental math is sixth and seventh grade. (I don't do much in eighth grade, and I do none in high school.) Mental math can take place during main lesson or in the afternoon math track class.

Mental math is a great way to get the students to focus at the start of class, and is an excellent tool for reviewing. Most of the students should be able to do the problems fairly easily on paper, but working them out completely in their head requires concentration. Complete silence in the classroom is essential. Avoid repeating questions, as this is also an exercise in listening. Students write the answer down, after finishing the problem in their head. All the students should be able to get the first couple of questions; the last few questions should be challenging. At the beginning of sixth grade there should be as many as 15 problems, each one fairly short. By the end of seventh grade, there should be fewer problems (perhaps only a total of 6), but they should be more challenging.

Another aspect of mental math, which the students love, is math tricks. I have only recently learned about these tricks, and I find it a shame that I was never taught them when I was in school. I have listed all of them in *Appendix B*. You will find some that you already know, albeit subconsciously. Others will surprise you.

Drill, repetition, and review!

Waldorf education seems to be the antithesis of dull, repetitive rote learning. We tend to be strong at introducing things in an interesting and imaginative way, but often there is not enough repetition and drill when it is most needed. The end result, frequently, is that our students don't retain what we've taught them – they can't remember how to do fractions; they don't have their multiplication facts down; and they are very slow at doing simple arithmetic. This, in my opinion, is a major weakness of many Waldorf schools.

It shouldn't be this way. Through mental math, homework, and in-class math sheets, the teacher needs to systematically plan how to integrate drill, repetition, and review into the routine in order to strengthen the students' skills. For the students who need more of this than others, the teacher needs to work with the extra-lesson teacher, the parents, or a tutor to ensure they get enough. Keep in mind that the time to start this is well before seventh grade.

Multiplication facts

If the students don't have their multiplication facts down by sixth grade, then the chances are that they will enter high school without knowing them. I now try to ensure that *all* my sixth grade students finish the year having their multiplication facts down cold. "Down cold" means that they know, for example, that 6 times 8 is 48 *instantly*, like they know their phone number. They should absolutely not have to think about it at all, and, especially, they shouldn't have to count on their fingers 8, 16, 24, 32, 40, 48 in order to arrive at it. This should be done in fourth or fifth grade. I always emphasize that memorizing the multiplication facts has nothing to do with how smart you are – you just have to memorize them. Reality is, however, that many students do poorly in math during their middle school and high school years partly because they lack confidence in math. This lack of confidence often starts from them not knowing their multiplication facts, which makes them *think* that they must be bad in math, and in the end it turns out to be a self-fulfilling prophecy.

So, what do I do to overcome this problem? Most importantly, I work with the parents. I emphasize how important it is that their child masters the multiplication facts. Many students already have it down, and those who believe they are really bad at it, usually only need to work on 5 or 10 facts, (e.g., 8x7=56; 9x6=54; 4x6=24). In such cases, I ask the parents to ensure that their child makes flashcards and works on them almost every day until they have been adequately learned. They should then review these every week until the end of the year.

Secondly, I do speed tests once per week, but only after I feel that the class has really learned their multiplication facts. Each sheet has about 100 problems. These include multiplication facts up to the 12's table, and two digit numbers plus or minus one digit numbers. All of it is in a random order. The students are given 6 minutes to get as far as they can. This helps make it more automatic. No counting on the fingers! It also increases their speed. I do not grade these speed tests, and it is not meant to be competitive. The point is to see their improvement as the year goes on. For copies of these speed tests contact Whole Spirit Press.

Calculators

The use of calculators is phased in slowly over three years, beginning in the eighth grade. The only time I permit the use of calculators in the eighth grade is at the end of the year with the units on *dimensional analysis* and on *growth* since calculations by hand can be quite tedious with these topics. In high school, calculators are used more and more each year.

Computers

I do a brief unit on computers in an effort to meet the needs of eighth graders who have a hunger for knowledge of the modern world as they prepare to enter high school. I do this without making it necessary for the students to be on a computer. I believe that actual work with computers should be delayed until high school. However, eighth graders should have a basic understanding of how computers work, and this understanding should be further developed in tenth and eleventh grade. Computers have two main functions: data storage (e.g., via a word processor, like WORD), and data manipulation (e.g., calculations, computer programs). In order to begin to understand about how computers store information, I do a unit on *number bases* and *ASCII code*, which gives a basic picture of how computer memory works. Many Waldorf schools incorporate *number bases* into their eighth grade curriculum.

One aspect of what I am proposing here – teaching about computer programming – is quite different from what is done in a typical Waldorf school. I call this unit *algorithms*; it is the "thinking" behind the computer. I have the students experience the thought process of computer programming without getting on a computer. Essentially, the students get to see algorithms, written in English, which are the equivalent of actual computer programs.

Should math classes be divided into faster and slower paced sections?

Many schools split the middle school afternoon math track classes according to ability level. The argument is that this improves the teacher-student ratio allowing struggling students to receive more individual attention. It is also argued that it allows the more advanced students to go faster.

I am personally opposed to splitting the math classes for several reasons.

Firstly, the math curriculum in the middle school, as outlined in this book, is generally not sequential. In other words, success in one topic does not depend on the success of a previous unit. This allows a student to recover from a unit in which he/she did poorly. Students that are really struggling should have tutoring outside school. High school, on the other hand, is a very different story. In ninth grade *algebra*, for example, each unit builds very sequentially upon the previous one. A student who fails the first quarter of ninth grade algebra will have a hard time catching up and surviving the year.

Secondly, and perhaps most importantly, if the classes were tracked starting in sixth or seventh grade, the students would be "put in a box" that would be difficult to get out of. This would be unfortunate given the fact that the middle school years are a time of great awakening in the students' thinking and some students "wake up" a bit later than others. I have seen many students who struggled at the start of seventh grade, but then "woke up", worked hard, and ended up entering high school very strong. If they had been placed in a slower-paced class in sixth or seventh grade, then that transformation would have been less of a possibility.

Thirdly, is the question: "Don’t we need a faster-paced class for the super-bright student that is bored being in the class with his/her slower classmates?" I have had many of these situations, and most of the time there are other issues at hand. Perhaps the parent wants their kid to be accelerated. Or maybe the student has no tolerance for doing repetition, even though they really need it. Or, what I have seen a lot of, is a student who really needs to work on social skills – in particular, developing patience for their fellow classmates. With most healthy situations, as long as I am doing my job well, the so-called super-bright kid is fine doing a few more fraction practice problems, and enthusiastically learns the new material at the pace that best suits the class, and is happy to be given the occasional challenge problem.

Having the whole class together is definitely more of a challenge for the teacher. It requires better classroom management skills, better organization, and a conscious effort to meet the needs of a more diverse class – but I feel it is well worth it!

Summary of Math Curriculum Topics

Sixth Grade

Arithmetic (75%)

The World of Numbers	Mental math & *math* tricks; casting out nines; exponents & roots; divisibility; prime factorization.
Division	Division and fractions; long division; why long division works; short division; checking answers.
Fractions	Thorough review; the relationship between fractions decimals & division; comparing fractions and decimals; compound fractions.
Decimals	Thorough review; converting between fractions and decimals; repeating decimals; converting repeating decimals to fractions.
Business Math &Percents	Introduction to percents; determining the percent of a given number; determining a percentage; percent increase and decrease; profit, commission & tax; simple interest; discount; loss; rate of pay; unit cost; temperature conversion formulas; business formulas; line graphs; pie charts.
Other Topics	Introduction to the metric system; word problems (rates); statistics; introduction to ratios; significant digits; currency exchange rates.

Geometry (25%)

General Concepts	Circle & polygon terminology; angle measure; the three dimensions.
The Basic Constructions	Copying a line segment; copying an angle; bisecting a line segment; bisecting an angle; construction of perpendicular lines; construction of a parallel line; division of a line into equal parts; construction of regular polygons (square, hexagon, etc.).
Spirals	Equiangular spirals; the Archimedean spiral.
Advanced Constructions	Rotations of circles; the limaçon and the cardioid; the hierarchy of quadrilaterals; knot and interpenetrating polygons; the 24-division with all its diagonals; the King's Crown.
Area	Areas of rectangles, squares, and right triangles.

Math Main Lesson Blocks

1. Business Math (including Percents, Formulas, and Graphing
2. Geometry (geometric drawing)

Afternoon Math Track Class meets twice per week.

Seventh Grade

Arithmetic (50%)

The World of Numbers	Mental math & math tricks; divisibility; roots.
Measurement	The metric system; review of the U.S. system.
Percents	Finding the base; strange percents; compound interest; calculating the percentage of increase or decrease.
Ratios	The three thoughts; the two forms; reciprocals of ratios ; proportion of the whole; similar figures; direct and inverse proportion.
Irrational Numbers	The ratio in a square; the ratio in a circle (π); repeating decimals; rational & irrational numbers; the square root algorithm.
Other Topics	Puzzle problems with doubling; word problems (rates).

Algebra (20%)

Basic Ideas	Basic goals; the importance of form; an introductory puzzle; history; terminology.
Negative Numbers	A careful introduction; combining positive & negative numbers; rules for multiplication & division.
Expressions	Simplifying expressions.
Formulas	Gauss's summing formula; car rental formula; Galileo's law of falling bodies; Euclid's perfect number formula.
Equations	An equation as a puzzle; solving equations by *Guess and Check*; the *Golden Rule of Equations*; solving equations by balancing.
Algebraic Word Problems	An introduction to algebraic word problems.

Geometry (30%)

Area	The shear and stretch; areas of parallelograms, trapezoids, and non-right triangles.
Geometric Drawing	Triangle constructions (SSS, SAS, ASA, SSA, AAS); the Greek geometric game; other methods; geometric division; star patterns;
The Pentagon & The Golden Ratio	Construction and properties of the pentagon; the golden ratio; the golden rectangle & golden spiral; the golden triangle.
Angle Theorems & Proofs	Theorems arising from two parallel lines cut by a transversal; angles in a triangle add to 180°; angles in other polygons; angle puzzles; Theorem of Morley; Theorem of Thales.
The Pythagorean Theorem	Visual proofs; Pythagorean triples; calculating missing sides of triangles.
Other Topics	Perspective drawing, various other drawing exercises.

Math Main Lesson Blocks

1. Algebra
2. Geometry (geometric drawing, areas, theorems up to the Pythagorean Theorem)

Afternoon Math Track Class meets three times per week.

Eighth Grade

Arithmetic (45%)

Number Bases	Ancient number systems; expanded decimal notation; scientific notation; base-8; base-5; base-16 (hexadecimal); base-2 (binary); arithmetic in various bases; converting between binary and hexadecimal.
The World of Numbers	Square root algorithm; Pythagorean Theorem.
Percents & Growth	Four ways to find the base; increase/decrease problems; exponential growth; the exponential growth formula; the rule of 72.
Dimensional Analysis	The two methods; Converting between metric and U.S. units; converting units for rates; converting areas and volumes; density.
Proportions	Shortcuts for solving (moving along diagonals, cross-multiplying); solving word problems with proportions; rate problems.

Algebra (10%)

Expressions	The laws of exponents; fractions & negatives.
Equations	Order of operations; evaluating expressions; distributive property; equations with fractions; strange solutions; converting repeating decimals into fractions.

Computers (5%)

Computer Memory & ASCII code	Bits and bytes; decoding binary codes.
Computer Algorithms	Writing algorithms using English; the prime number algorithm; an algorithm for addition; an algorithm for long division, the square root algorithm.

Geometry (40%)

Mensuration	Baravalle's proof of the Pythagorean Theorem; area of a trapezoid; Heron's formula; the area of four types of triangles; area of a circle; portions of circles; volume & surface area of solids (box, prism, pyramid, cylinder, cone, sphere, octahedron, tetrahedron); Archimedes' ratio; tricks with dimensions.
Stereometry	Types of polyhedra; Platonic solids; the transformation of solids; orthogonal views; duality; Archimedean solids; the stretching process; the Archimedean duals; constructing paper model; close-packing; Euler's formula; imagination 3-D transformation exercises.
Loci	Curves generated form loci problems (a circle, two parallel lines, two concentric circles, a perpendicular bisector, two angle bisectors, parabola, ellipse, hyperbola); alternative definitions; conic sections; curves in movement, the Curves of Cassini.

Math Main Lesson Blocks

1. Number bases and Loci.
2. Geometry (mensuration and stereometry)

Afternoon Math Track Class meets three times per week.

Sixth Grade

The year for strengthening skills

While there are new topics to be introduced in sixth grade math, much of the year is an important review, or a furthering of material introduced in earlier years. The challenge is to weave in the review in such a way that there is always something new. Each homework or practice sheet should include a fair amount of review problems. If your students enter seventh grade feeling that division, fractions, and decimals are all "easy", and they are excited about learning math, then you have succeeded.

A continual theme through the year is the sense of number and the interrelationship between division, fractions, decimals, and percents. Fractions play the central role. The key is that division, decimals, and percents can all be thought of as a fraction.

Another theme in sixth grade math is developing good work habits. I believe that weekly homework should be assigned and that it should be unthinkable that a student wouldn't complete it. Organization skills, including a good notebook, are very important.

The order of topics

The order in which topics are introduced in my workbook is as follows:

> The four processes with fractions and decimals, long division (including repeating decimals), reducing fractions, casting out nines, short division, mixed numbers and improper fractions, exponents, converting decimals to/from fractions, estimating, square roots, divisibility, unit cost, U.S. measurement, formulas, metric, converting repeating decimals to fractions, factors and prime numbers, angle measurement, prime factorization, basic percents, mean/median/mode, pie charts, area and perimeter, percent increase and decrease, tax rate, discount, profit and loss, rate of pay, converting to/from percents, ratios, rate of speed, line graphs, foreign exchange rates, compound fractions.

Arithmetic

The World of Numbers

Mental Math

- See sections on *Mental Math* and *Multiplication Facts* in the *Introduction*.
- *Begin every class,* throughout the whole year, with either a speed test or mental math.

Math Tricks

- Cover the sixth grade math tricks. (**See Appendix B.**)
- Introduce one every other week and keep the tricks fresh by including them in mental math throughout the year.
- In general, the idea in sixth grade is not so much to explain why each math trick works, but instead to use them to build the students' calculating skills and to increase their confidence. These tricks will also develop their sense of wonder for numbers.

New multiplication facts to be memorized ("*" indicates optional)**:**

13x2 = 26	14x2 = 28	15x2 = 30	16x2 = 32	18x2 = 36	25x2 = 50
13x3 = 39	14x3 = 42	15x3 = 45	16x3 = 48	*18x3 = 54	25x3 = 75
13x4 = 52	*14x4 = 56	15x4 = 60	16x4 = 64	*18x4 = 72	25x4= 100
*13x5 = 65	*14x5 = 70	15x5 = 75	*16x5 = 80	*18x5 = 90	25x5 = 125
13x13 = 169	14x14 = 196	15x15 = 225	16x16 = 256	18x18 = 324	25x6 = 150
					*25x8 = 200
					25x25 = 625

Casting Out Nines

- A must do! Lots of fun!
- Normally, we check a multiplication problem to see if it is right simply by redoing the problem. This is problematic for two reasons**:** it is time consuming, and we are likely to make the same mistake again.

- *Casting out nines* allows us to quickly check our answer after doing a multiplication problem.

Example: The key is to realize that the *arrows represent summing the digits* (e.g., with 7296**:** 7+2+9+6=24)**:**

```
  7296 ———→ 24 ——→    6
 x 376 ———→ 16 ——→  x 7
 43776               42 ↘ (6)
51072                     Correct!
21888
2743296 ———→ 33 ———→ (6)
```

If the circled results aren't the same, then there is a mistake in the multiplication. A short cut for summing the digits is to *cast out* all groups of digits that add to *nine*, or multiples of nine. Thus, with the answer 2743296**:** the first two digits (27), the next three (432), and then the 9, are all *cast out*, leaving just the 6 as the result. With practice, this is very quick!

Exponents and Roots

- Introduce exponents and roots using only numbers and simple examples. Do *not* use variables.

Example: $3^4 = 3{\cdot}3{\cdot}3{\cdot}3 = 81$
Example: $10^3 = 1000$
Example: $2^5 = 32$
Example: $(0.02)^3 = 0.000008$

Example: $\left(\frac{2}{5}\right)^2 = \frac{4}{25}$
Example: $(4\tfrac{1}{2})^2 = \left(\frac{9}{2}\right)^2 = \frac{81}{4} = 20\tfrac{1}{4}$

Example: $\sqrt{25} = 5$
Example: $\sqrt{1000000} = 1000$
Example: $\sqrt{12100} = 110$
Example: $\sqrt{160000} = 400$

- Have the students calculate the *powers of two* as high as they can go. You may want to have them check their answer with every exponent increase of 10 or 20. (See **Appendix E,** *Powers of Two Table* for a listing of the powers of two up to 2^{100}.)
- The students should memorize the following powers**:**
  - $2^3 = 8$; $2^4 = 16$; $2^5 = 32$; $2^6 = 64$; $2^{10} = 1024$; $3^3 = 27$; $3^4 = 81$; $4^3 = 64$; $4^4 = 256$; $4^5 = 1024$; $5^3 = 125$; $5^4 = 625$
  - Optional ones: $2^7 = 128$; $2^8 = 256$; $2^9 = 512$; $3^5 = 243$; $3^6 = 729$; $6^3 = 216$; $7^3 = 343$; $8^3 = 512$; $9^3 = 729$

Divisibility Rules

- A number is evenly divisible[1] by 2 only if it is even.
- A number is evenly divisible by 3 only if the sum of the digits is divisible by 3. The nice thing here is that we can *cast out threes* or groups of digits adding to multiples of three (3, 6, 9, 12, etc.). For example, with 65387 we can immediately cast out the 6 and 3 because they are divisible by 3, and then we can cast out the 8 and 7 because they add to 15. This leaves us with just the 5, which is not divisible by 3, so *we conclude that 65387 is not evenly divisible by 3*.
- A number is evenly divisible by 4 only if the last two digits are divisible by 4. For example, 6380716 is evenly divisible by 4, because it ends in 16, which is evenly divisible by 4.
- A number is evenly divisible by 5 only if the number ends in a 5 or a 0.
- A number is evenly divisible by 9 only if the sum of the digits is divisible by 9. Again we can *cast out nines* in order to check divisibility for 9 quickly. If we cast out nines and are left with nothing in the end, then the number is evenly divisible by nine. For example, for 71,284 we cast out the 7 and 2 and then cast out the 8 and 1 and we are left with just a 4, so the whole number is not evenly divisible by nine.
  On the other hand, with 2,381,697 we cast out the 8 and 1, the 6 and 3, the 2 and 7, and the 9, leaving us with nothing. Therefore, we can conclude that 2,381,697 is evenly divisible by nine.
- A number is evenly divisible by 10 only if the number ends in a 0.
- *Practice using the divisibility rules to reduce large fractions.*

Example: Reduce $\frac{132}{420}$

Solution: We recognize that both the denominator and numerator are evenly divisible by 4 and 3. So after dividing both the denominator and numerator by 4 and 3 we get an answer of $\frac{11}{35}$.

Example: Reduce $\frac{54}{126}$

Solution: Dividing both the denominator and numerator by 2 and 9 gives an answer of $\frac{3}{7}$.

Example: Reduce $\frac{14175}{14850}$

Solution: Dividing both the denominator and numerator by 9 gives us $\frac{1575}{1650}$, then dividing by 5 gives $\frac{315}{330}$, then dividing by 5 again gives $\frac{63}{66}$, and then finally dividing by 3 gives our answer of $\frac{21}{22}$.

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[1] "Evenly divisible" means it can be divided with no remainder.

Prime Factorization

- *Don't use factor trees*. While factor trees work, students often don't understand them, and are puzzled about what the final prime factorization is by looking at the tree.
- The best method is to keep breaking down any non-prime number into the product of two more numbers until there are only prime numbers left. The students need to realize that each step in the process is equal.

  Example: Find the prime factorization of 700.

  Solution: There are several routes to the answer. Below, we show two different ways to arrive at the same answer. Remember that each step represents a different way to express 700 as a product of numbers.

  - $700 \rightarrow 7{\cdot}100 \rightarrow 7{\cdot}2{\cdot}50 \rightarrow 7{\cdot}2{\cdot}5{\cdot}10 \rightarrow 7{\cdot}2{\cdot}5{\cdot}5{\cdot}2 \rightarrow 2{\cdot}2{\cdot}5{\cdot}5{\cdot}7 \rightarrow \underline{2^2 \cdot 5^2 \cdot 7}$
  - $700 \rightarrow 25{\cdot}28 \rightarrow 5{\cdot}5{\cdot}28 \rightarrow 5{\cdot}5{\cdot}14{\cdot}2 \rightarrow 5{\cdot}5{\cdot}7{\cdot}2{\cdot}2 \rightarrow 2{\cdot}2{\cdot}5{\cdot}5{\cdot}7 \rightarrow \underline{2^2 \cdot 5^2 \cdot 7}$

  Example: Find the prime factorization of 208. (Answer**:** $2^4 \cdot 13$)

  Example: Find the prime factorization of 12375. (Answer**:** $3^2 \cdot 5^3 \cdot 11$)

- (optional) *Least common multiples* (LCM) and *greatest common factors* (GCF).
  - Simple cases can easily be done in your head, such as**:**

    Example: Find the LCM and GCF of 12 and 8.

    Solution: The LCM is <u>24</u> and the GCF is <u>4</u>.
  - For larger numbers, it is useful to use prime factorization in order to determine LCMs and GCFs. Build up to problems like**:**

    Example: Find the LCM and GCF of 29,040 and 207,900.

    Solution: The prime factorization for 29,040 is $2^4{\cdot}3{\cdot}5{\cdot}11^2$, and for 207,900 is $2^2{\cdot}3^3{\cdot}5^2{\cdot}7{\cdot}11$. The GCF is what is in common for both, therefore**:** $2^2{\cdot}3{\cdot}5{\cdot}11$, which is <u>660</u>. The LCM includes everything from both (without duplicating), therefore**:** $2^4{\cdot}3^3{\cdot}5^2{\cdot}7{\cdot}11^2$, which is <u>9,147,600</u>.
  - *Common denominators* for "ugly" fractions.

    Example: $\frac{5}{3024} + \frac{11}{576}$

    Solution: The prime factorization of 3024 is $2^4{\cdot}3^3{\cdot}7$ and the prime factorization of 576 is $2^6{\cdot}3^2$, therefore the common denominator (LCM) is $2^6{\cdot}3^3{\cdot}7$ (which is 12096). The first fraction must have its denominator and numerator multiplied by 2^2 (or 4), and the second fraction by $3{\cdot}7$ (or 21). This gives an answer of $\frac{5\cdot4}{12096} + \frac{11\cdot21}{12096} = \frac{251}{12096}$.

Division

Division and Fractions

- *Think of division as a fraction*
  - Reduce before dividing. (**See Appendix B,** *6th Grade Math Tricks.*)

    Example: 804÷44

    Solution: 804÷44 becomes $\frac{804}{44}$ and is then reduced to $\frac{201}{11}$. The division problem has been made easier. We now can divide 201 by 11, to get an answer of <u>$18\,{}^3/_{11}$</u>.

    Example: 108÷48 is reduced (by dividing top and bottom by 12) to 9÷4, which is <u>2¼</u>.
  - *Making the divisor easier.* Think Fractions! There are two ways to make the divisor easier**:**
    - Move the divisor's decimal point all the way to the right.

      Example: 30.17÷0.035 becomes 30170÷35 since we multiply numerator and denominator by 1000, or move the decimal point three places.
    - Chop off ending zeroes by moving the decimal point, which is initially invisible, to the left.

      Example: 173.6÷800 becomes 1.736÷8 since we divide numerator and denominator by 100, which is the same as moving the decimal point two places to the left.

Long Division

- *Vocabulary*
  - The *dividend* is the number that is being divided by the *divisor*. The *quotient* is the answer.

```
               324  ← Quotient
Divisor  →  5 |1620 ← Dividend
```